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NEW GUIDELINES FOR OPTIMIZATION OF FINITE ELEMENT SOLUTIONS

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Abstract

In the optimization of finite element solutions, an algorithmic procedure is required to generate a finite element mesh that achieves the desired accuracy with minimal computational effort. Traditionally, two approaches—analytical and topological guidelines—have been employed for this purpose. However, the effectiveness of the guidelines derived from these approaches has not been systematically evaluated. In this study, a novel parameter is introduced to formulate a proposed optimality condition. Based on this parameter and the contours of strain energy density, new mesh optimization guidelines are developed. The results demonstrate that the proposed guidelines yield superior solutions compared with those obtained using existing methods.

Keywords: optimization, finite element mesh, strain energy density, guidelines.

Introduction

Shephard (1979) defined the optimization of finite element solutions as an algorithmic procedure for generating a finite element discretization that achieves the required accuracy with minimal computational effort. Two principal approaches to finite element optimization have been identified: analytical and topological.

The analytical approach treats nodal coordinates as unknowns in the formulation of the potential energy functional. This approach involves two major challenges: (1) the strong nonlinearity of the governing equations and () the presence of nonlinear constraints on the nodal coordinates. Together, these factors render the solution process computationally demanding and time-consuming.

In contrast, topological approaches focus on optimizing mesh configurations and have led to the development of practical guidelines that assist analysts in constructing grid layouts whose mesh topology closely approximates that of an optimal mesh for a given problem (Turcke, 1974). Compared with analytical methods, topological approaches generally require less computational effort; comparable accuracy can often be achieved at a lower computational cost by combining topological guidelines with mesh refinement techniques.

Over the past two decades, extensive research has been devoted to topological mesh optimization. Various mesh optimization criteria have been proposed by Oliveira (1973), Turcke and McNeice (1974), Shephard, Gallagher, and Abel (1979), and Turcke (1974). In addition, Turcke and McNeice (1974) developed practical mesh optimization guidelines. Despite these contributions, the literature still lacks a definitive and universally accepted method for evaluating the quality of an "optimal" finite element solution. As a result, different researchers employing different optimization methodologies often obtain distinct "optimal" mesh configurations for the same problem.

In this study, an optimality condition for finite element solutions is proposed for the specific problem under consideration. A novel parameter, termed the degree of freedom density (DOFD), is introduced and employed to formulate this optimality condition. Based on the DOFD and the contours of strain energy density (SED), new mesh optimization guidelines are developed.

To assess the effectiveness of the proposed guidelines, a square plate subjected to concentrated loads at its four corners is analyzed. The results demonstrate that the proposed mesh optimization guidelines yield more effective solutions than those obtained using currently available guidelines.

Literature Review

Analytical Investigation

To provide the necessary theoretical background and establish the context for the subsequent analysis, several fundamental mathematical concepts are briefly reviewed. As noted by Turcke and McNeice (1974), the total potential energy of the system can be expressed as follows:

$$\pi_p = \int_V \frac{1}{2} \underline{u}^T \underline{\underline{B}}^T \underline{\underline{D}}\,\underline{\underline{B}}\,\underline{u}\, dV - \int_V \underline{X}^T \underline{u}\, dV - \int_{S_t} \underline{P}^T \underline{u}\, dS_t \tag{1}$$

where

$\underline{u}$ = the displacement vector;
$\underline{\underline{B}}$ = the displacement-strain matrix;
$\underline{\underline{D}}$ = the stress-strain matrix;
$\underline{X}$ = the body force density vector;
$\underline{P}$ = the surface traction vector;
S_t = a boundary over which surface tractions are specified.

Since the displacement–strain matrix $\underline{\underline{B}}$ is obtained by differentiating the element shape functions, which themselves are functions of the nodal coordinates x_j, the total potential energy can therefore be considered as a function of both the nodal displacements u_i and the nodal coordinates x_j.

$$\pi_p = \sum_{e=1}^{NUMEL} \pi_p^e (u_i, x_j) \tag{2}$$

where

$NUMEL$ = the total number of elements,
i = 1, 2, 3, ..., n,
j = 1, 2, 3, ..., m,
n = the number of unrestrained nodal displacements,
m = the number of unconstrained nodal coordinates.

For a system in a stationary state, the following conditions hold:

$$\frac{\partial \pi_p}{\partial u_i} = 0 \tag{3}$$

$$\frac{\partial \pi_p}{\partial x_j} = 0 \tag{4}$$

The corresponding equilibrium equations are obtained from Equations (3) and (4):

$$\underline{\underline{K}}\,\underline{u} - \underline{F} = \underline{0} \tag{5}$$

where $\underline{F}$ and $\underline{\underline{K}}$ denote the global load vector and the global stiffness matrix, respectively. Equations (3) and (4) then yield the following nonlinear equations:

$$\frac{1}{2}\underline{u}^{T}\frac{\partial \underline{\underline{K}}}{\partial x_j} - \frac{\partial \underline{F}}{\partial x_j}\underline{u}^{T} = \underline{0} \tag{6}$$

Neglecting the second term in Equation (6), as suggested by Melosh and Marcal (1977), leads to incorrect results. The global load vector $\underline{F}$ comprises contributions from concentrated loads, body forces, surface tractions, and loads induced by temperature changes. The evaluation of body forces and surface tractions requires volume and surface integrals, respectively, which involve the element shape functions. Consequently, the global load vector inherently depends on the nodal coordinates. Therefore, the second term in Equation (6) should not be omitted, except in cases where body forces and surface tractions are absent.

A sufficient condition for the solutions of Equations (5) and (6) to correspond to a minimum is that the Hessian matrix be positive definite. This condition is satisfied if all principal minors of the Hessian matrix are positive (Turcke and McNeice, 1974). The Hessian matrix is defined as follows:

$$\begin{bmatrix} \dfrac{\partial^2 \pi_p}{\partial x_K \partial x_l} & \dfrac{\partial^2 \pi_p}{\partial x_K \partial u_l} \\ \dfrac{\partial^2 \pi_p}{\partial u_K \partial x_l} & \dfrac{\partial^2 \pi_p}{\partial u_K \partial u_l} \end{bmatrix} \tag{7}$$

It should be noted that $\partial^2 \pi_p / \partial u_K \partial u_l$ corresponds to the stiffness matrix $\underline{\underline{K}}$. Therefore, for a given set of nodal coordinates, the conditions required for mesh optimization are identical to those necessary to ensure the uniqueness of the solution.

Topological Investigation

Two types of topological investigations are considered in this study: (a) the contouring method, and (b) mesh optimization based on the guidelines proposed by Turcke and McNeice (1974).

The Contouring Method

The contouring method utilizes contours of a selected solution parameter obtained from the current step (Turcke (1974, 1974a) to generate a new mesh for the subsequent step. Its primary advantage is that the resulting mesh is independent of the previous one, allowing degrees of freedom (DOFs) to be concentrated in regions where the solution parameter exhibits rapid spatial variation. However, the quantitative redistribution of DOFs remains challenging due to the absence of explicit guidelines.

Mesh Optimization Based on Turcke and McNeice Guidelines

In 1974, Turcke and McNeice proposed a set of guidelines to complement the contouring method. The procedure can be illustrated using a square plate subjected to loads at its four corners:

1. Starting from the initial uniform mesh (Figure 1), variations in strain energy density along selected paths are plotted, as shown in Figures 2 and 3.
2. To generate a new mesh, the number of contours, N, is first determined. A series of N+1 straight lines is then used to approximate the curves shown in Figures 2 and 3.
3. The intersections of these line segments define nodal points along specific paths on the plate and indicate the approximate locations of the isoenergetic contours (Figure 4).
4. In regions with high strain energy gradients, nodes are initially placed along the isoenergetic contour with the highest values. Although additional nodes could be added along this contour, doing so would increase the degrees of freedom (DOFs) beyond those of the current mesh.
5. After establishing elements along the highest contour level and the plate boundary, subsequent elements are generated to extend from this region between successive isoenergetic lines, following trajectories normal to the contours.
6. The resulting mesh configuration is shown in Figure 5. The advantage of this method lies in its straightforward procedural approach. Its main limitation is the difficulty in assessing whether the series of N+1 straight lines sufficiently captures the key features necessary for an optimal mesh design.

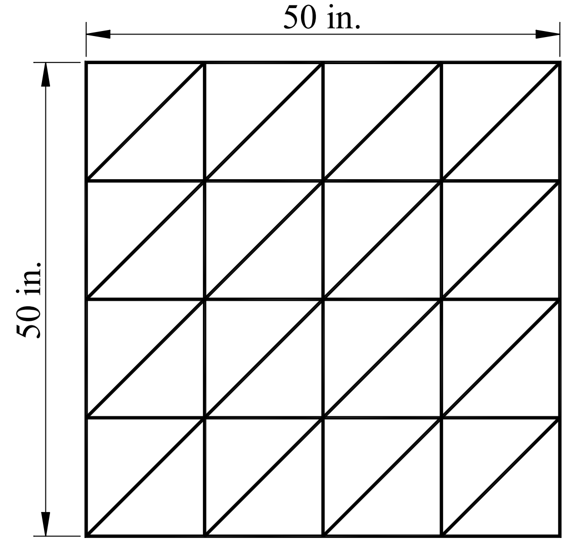


Figure 1. Uniform mesh configuration proposed by Turcke and McNeice (1974)

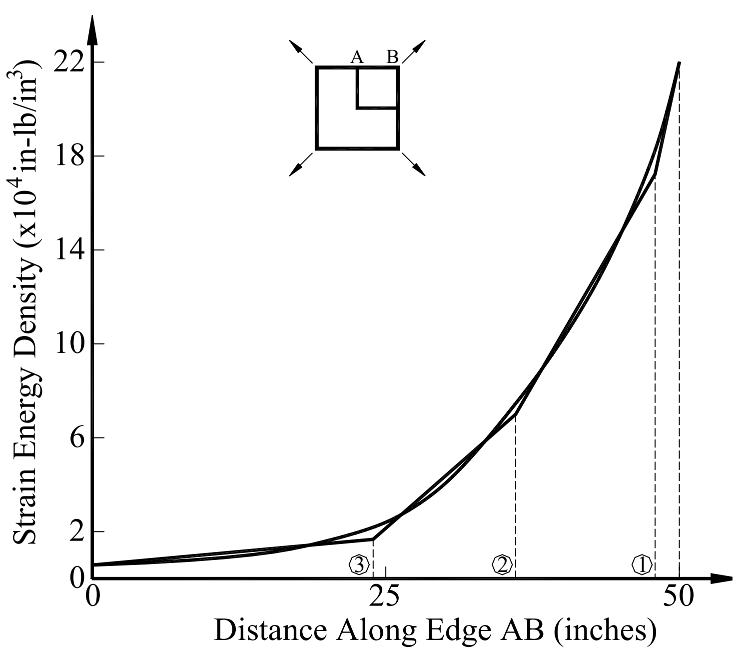


Figure 2. Variation of strain energy density along edge AB, computed using the mesh configuration proposed by Turcke and McNeice (1974)

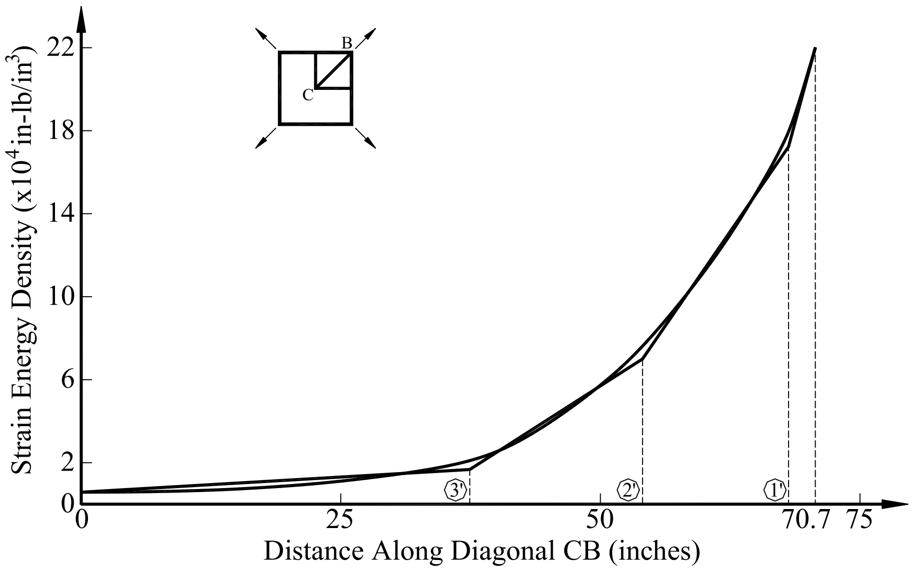


Figure 3. Variation of strain energy density along diagonal CB, computed using the mesh configuration proposed by Turcke and McNeice (1974)

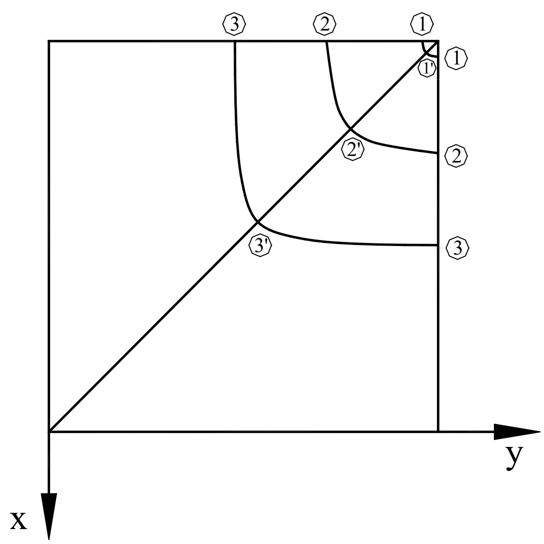


Figure 4. Contours of strain energy density computed using the optimal mesh configuration proposed by Turcke and McNeice (1974).

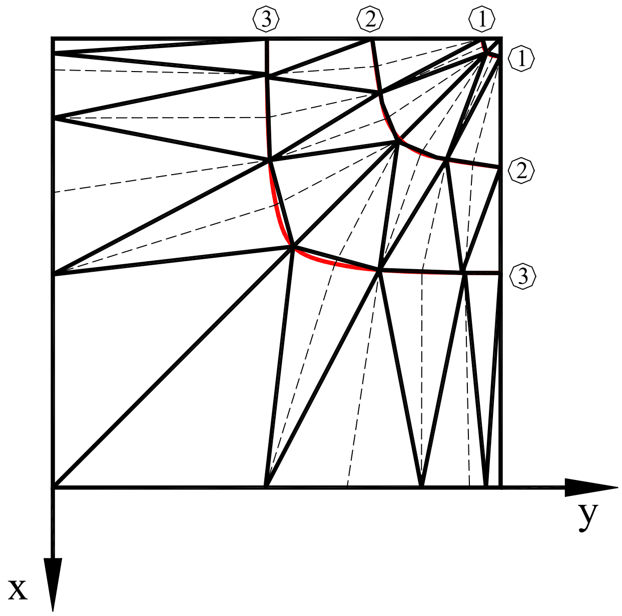


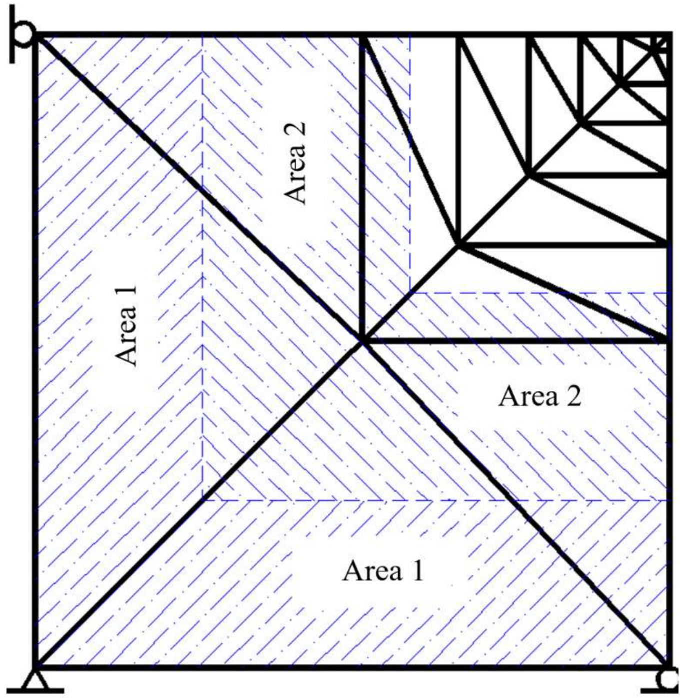
Figure 5. Configuration of the optimal mesh proposed by Turcke and McNeice (1974_

Proposed Guidelines

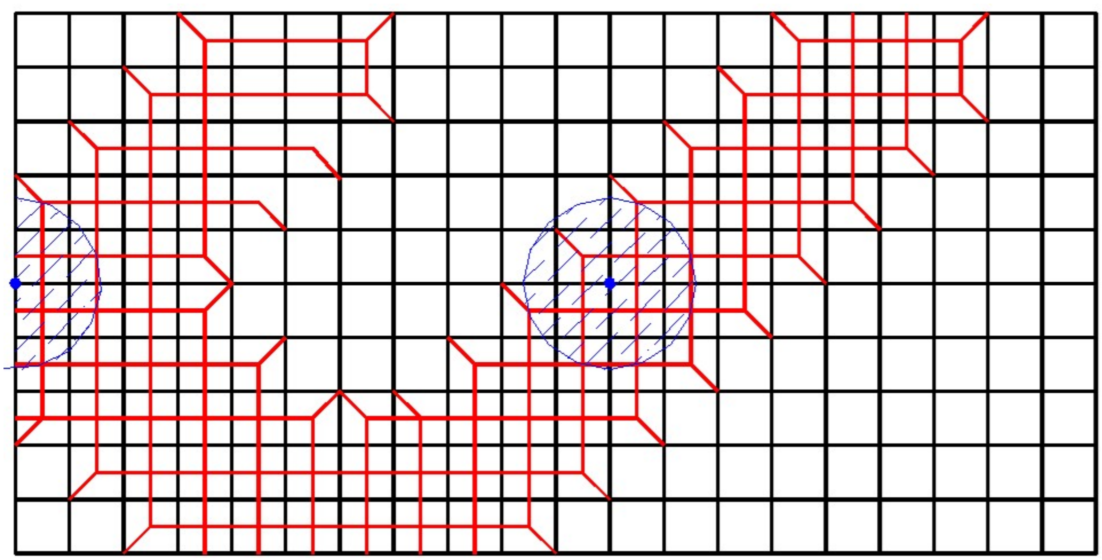
First, the degree of freedom density (DOFD) is defined. Next, the conditions for obtaining optimal finite element solutions are established. Finally, based on these two steps, new guidelines for mesh optimization are proposed.

Degree of Freedom Density (DOFD)

The degree of freedom density (DOFD) is defined as the ratio of the total number of degrees of freedom within a given area to the size of that area. For simple problems, such as the one illustrated in Figure 6(a), the predefined areas (1, 2, etc.) can be used directly to calculate the DOFD. For more complex problems, such as the one shown in Figure 6(b), a moving circle or square of constant area can be employed as the defined region. By centering the moving circle or square on each nodal point, the total number of degrees of freedom within the region can be determined, thereby allowing the DOFD to be calculated. If a nodal point is located near the mesh boundaries, only the portion of the region that lies within the mesh should be considered.



a. The defined area in a simple mesh



b. The defined area in a complicated mesh

Figure 6. Definition of the area used in the determination of the degree of freedom density

Conditions for Optimal Finite Element Solutions

When the variational approach is employed in finite element methods, the resulting total strain energy is generally less than that of the exact solution for a given load. For complex problems, the exact total strain energy cannot be determined, making it difficult to assess the quality of finite element solutions. By comparing the total strain energies obtained from different mesh optimization methods (Shephard, Gallagher, and Abel, 1979), it is evident that the mesh with the highest total strain energy is not necessarily the optimal one. In contrast, comparing the total strain energy of an optimized mesh with that of a very fine mesh (Turcke and McNeice, 1974) provides a relatively straightforward means of distinguishing between solutions. Since the goal of mesh optimization is to achieve the highest possible accuracy with minimal computational effort, using an extremely fine mesh often constitutes over-refinement.

Given these challenges, it is important to establish a condition based on solution parameters obtained at the current step to assess the quality of finite element solutions. Accordingly, the condition for optimal finite element solutions can be stated as follows:

If the variations of the strain energy density (SED) and the degree of freedom density (DOFD) along any selected path in a mesh are plotted together, the solution is considered optimal when the SED curve coincides with the DOFD curve.

Optimization of Finite Element Solutions

Based on the two concepts introduced above, the following guidelines for mesh optimization are proposed:

1. Initial Mesh Selection: For a given problem, basic engineering judgment can be used to allocate more degrees of freedom (DOFs) in regions where stress concentrations are anticipated. Otherwise, a uniform initial mesh may be employed.
2. Plotting SED and DOFD: Using the solution obtained from the initial mesh, variations of the strain energy density (SED) and the degree of freedom density (DOFD) along selected paths are plotted.
3. Assessing Solution Quality: By comparing the SED and DOFD curves, the analyst can assess the quality of the finite element solution.
4. Identifying the Optimal Solution: The optimal solution is achieved when the SED curve closely coincides with the DOFD curve, or when the total strain energy exhibits minimal variation with changes in the length ratio of successive segments along a selected path. Once these conditions are satisfied, further iterations can be terminated.
5. Refining the Mesh: If further iterations are required, the length ratio between successive segments along each selected path should be adjusted. Before doing so, the number of segments, N, for each path must be determined. As a general rule, in

creasing the number of segments improves solution accuracy. The length ratio is then determined by examining the gradient of the SED so that the variation of the DOFD follows that of the SED. Each selected path is divided proportionally into N segments, and the nodal locations along the path are established accordingly.

6. Generating a New Mesh: A new mesh is created using the updated nodal points.
7. Iterative Procedure: By repeating steps 2–6, the analyst can approach the optimal solution within a few iterations.

Numerical Analysis

The problem analyzed in this section involves a square plate subjected to diagonal loads at its four corners, a configuration commonly used in previous studies. Due to axis-symmetry, only one-quarter of the plate is modeled. The material properties for this problem are specified as follows:

- Young's modulus: E=10,000,000 psi
- Poisson's ratio: ν=0
- Thickness: t=1.0 inch
- Length: L=100inch
- Applied load: P=25,000,000 lbs

The 'optimal' meshes reported in the literature are first examined. Then, using the newly proposed guidelines, a mesh that closely approximates the optimal finite element configuration is generated. Finally, all results are compared and discussed.

Turcke and McNeice's Guidelines

The optimal mesh obtained using Turcke and McNeice's guidelines is shown in Figure 3.5. The total strain energy was found to be [value missing]. The maximum displacements in both the X and Y directions are 6.392 inches. The variations of strain energy density (SED) and degree of freedom density (DOFD) along edge AB and diagonal CB are presented in Figures 7 and 8, respectively. From these figures, it is evident that a significantly higher concentration of degrees of freedom is required in regions with high strain energy density gradients. Failure to adequately allocate DOFs in these regions leads to reduced solution quality.

Turcke's Mathematical Programming Technique

The optimal mesh obtained using Turcke's mathematical programming technique is shown in Figure 9. The total strain energy was found to be [value missing]. The maximum displacements in both the X and Y directions are 6.649 inches. The variations of strain energy density (SED) and degree of freedom density (DOFD) along edge AB and diagonal CB are presented in Figures 10 and 11, respectively.

From these figures, it is evident that the DOFD curves lie below the SED curves, indicating that the available degrees of freedom are insufficient throughout the domain. Consequently, although the curves conform better in regions with high strain energy density gradients compared to the previous case, the overall improvement is minimal. Therefore, the solution quality remains poor.

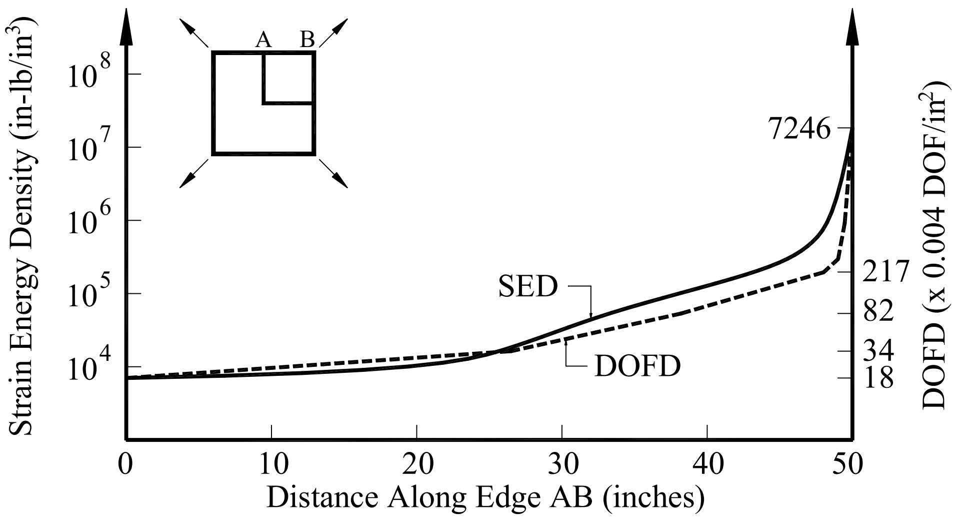


Figure 7. Strain energy density and degree of freedom density distributions along edge AB.

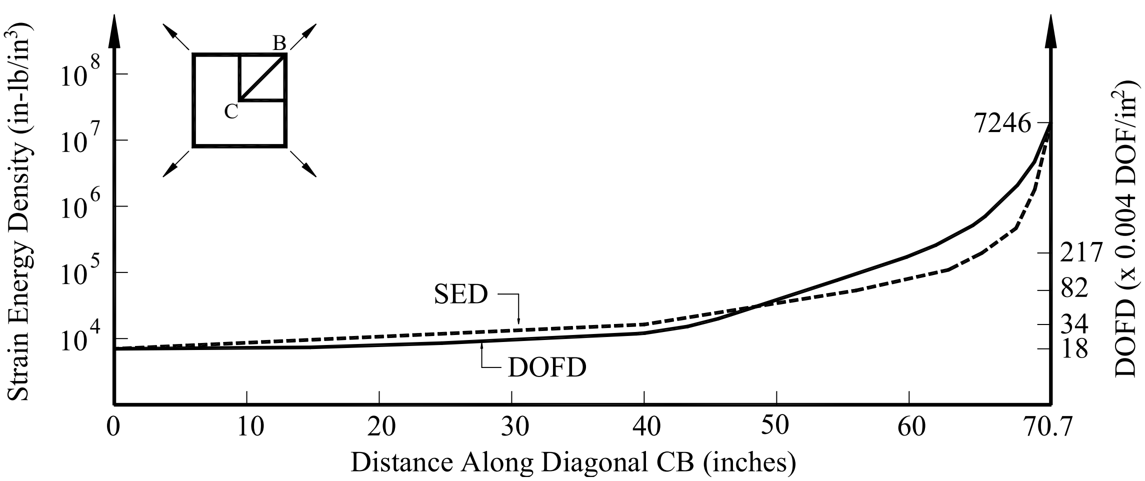


Figure 8. Strain energy density and degree of freedom density distributions along diagonal CB.

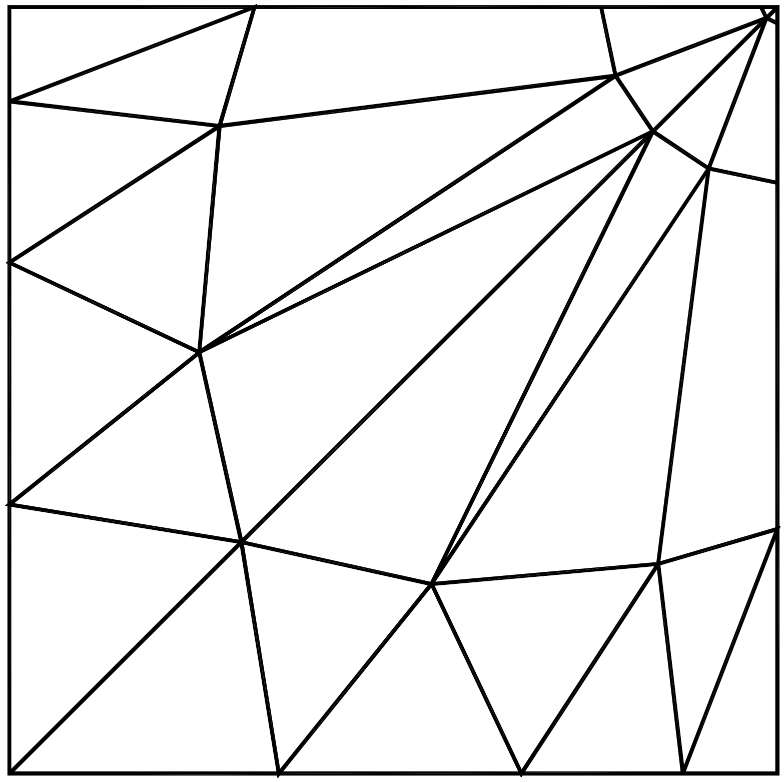


Figure 9. Optimum mesh obtained through Turcke's mathematical programming technique (1979

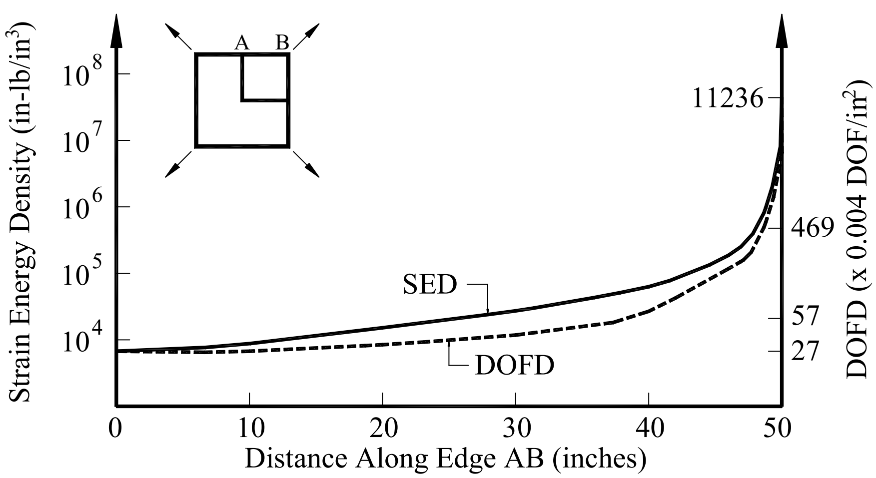


Figure 10. Strain energy density and degree of freedom density distributions along edge AB.

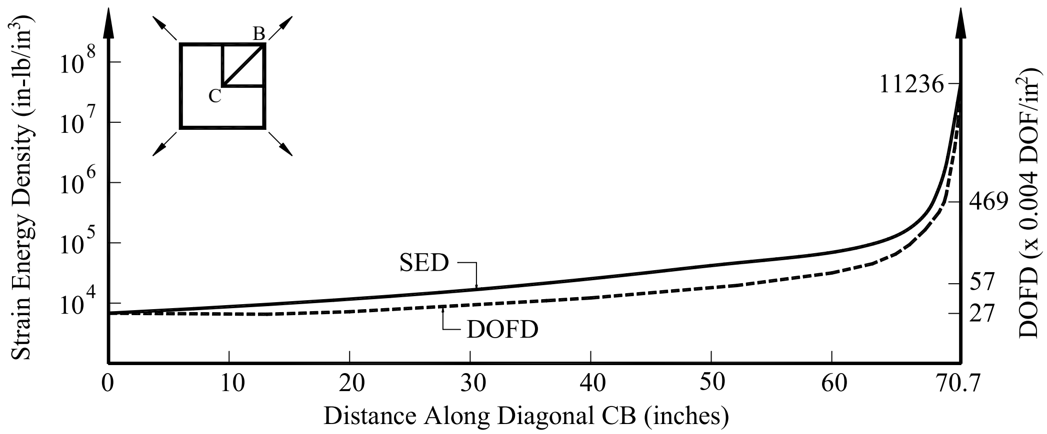


Figure 11. Strain energy density and degree of freedom density distributions along diagonal CB.

Shephard's Contouring Method

The optimum mesh obtained using Shephard's contouring method (Shephard, 1979) is shown in Figure 12. The total strain energy was found to be [value missing]. The maximum displacements in both the x and y directions are 7.167 inches. The variations of strain energy density and degree of freedom density along edge AB and diagonal CB are presented in Figures 13 and 14, respectively.

From these figures, it can be observed that, except in regions with high strain energy density gradients, the degree of freedom density curves lie above the strain energy density curves. This indicates that the solutions obtained using Shephard's contouring method (Shephard, 1979) are generally better than those of the previous two methods. However, in regions with high strain energy density gradients, the curves do not match the strain energy density very well. Therefore, the overall solution quality may still be insufficient.

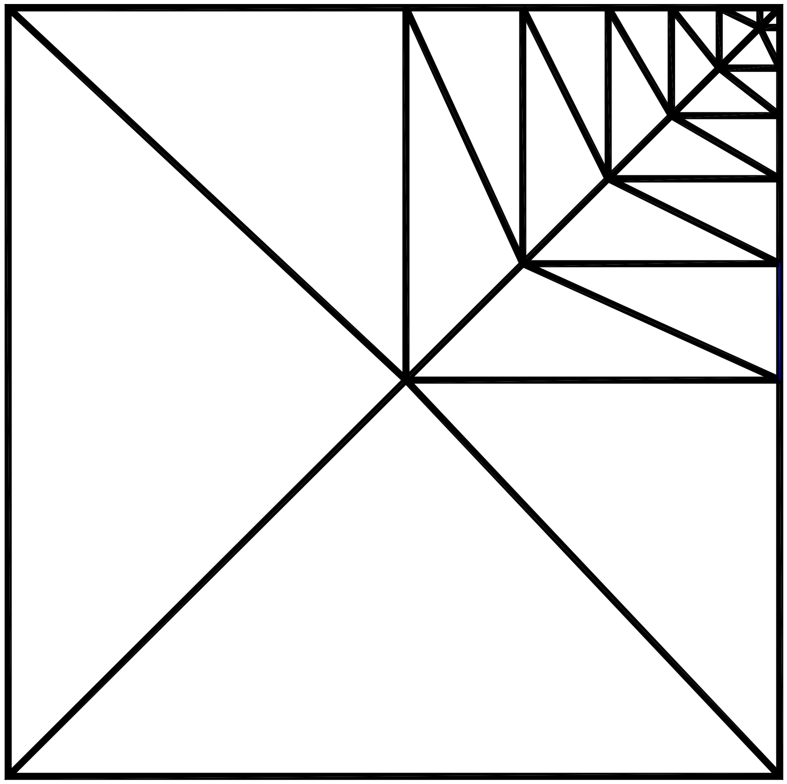


Figure 12. Optimum mesh obtained through the contouring method proposed by Shephard (1979).

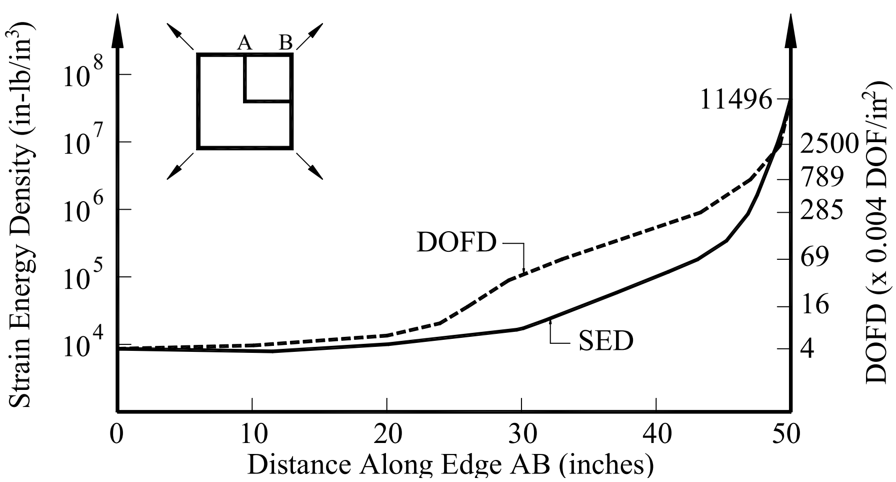


Figure 13. Strain energy density and degree of freedom density along edge AB using Shephard's contouring method (Shephard, 1979).

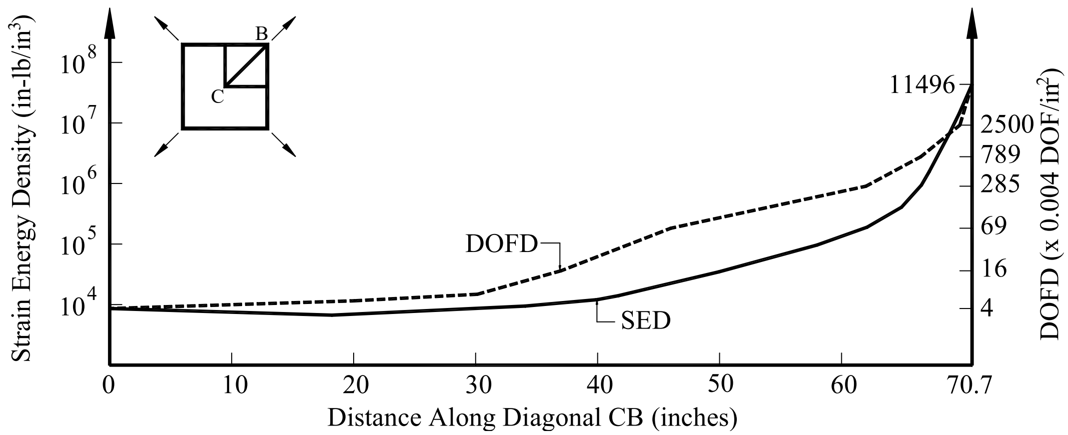


Figure 14. Strain energy density and degree of freedom density along diagonal CB using Shephard's contouring method (Shephard, 1979).

New Proposed Guidelines by the Authors

The uniform mesh shown in Figure 1 is first examined. The variations of strain energy density and degree of freedom density along edge AB and diagonal CB are presented in Figures 15 and 16, respectively. From these figures, it is evident that the solution quality is extremely poor, as the degree of freedom density curves diverge from the strain energy density curves in regions with high strain energy density gradients. This indicates that a mesh with a length ratio equal to 1 produces very poor results.

The length ratio is defined as the ratio of the segment length on the lower strain energy density side to that on the higher strain energy density side. By varying the length ratio, the solutions corresponding to the new mesh can be obtained. Plotting the total strain energy versus the length ratio along the selected paths, as shown in Figure 3.17, the maximum strain energy is found to be [value missing], with a corresponding length ratio of 3.5826. The lengths of the segments along the selected paths AB and CB are listed in Table 1.

The optimum mesh based on the proposed guidelines is shown in Figure 18. The variations of strain energy density and degree of freedom density along edge AB and diagonal CB are shown in Figures 19 and 20, respectively. From Figure 19, it can be seen that the degree of freedom density curve closely matches the strain energy density curve along edge AB. From Figure 20, the degree of freedom density curve aligns well with the strain energy density curve in regions with high strain energy density gradients. Although the fit is not as precise in regions with low strain energy density, the effect is minimal. Therefore, the overall quality of the solutions is considered reasonably good.

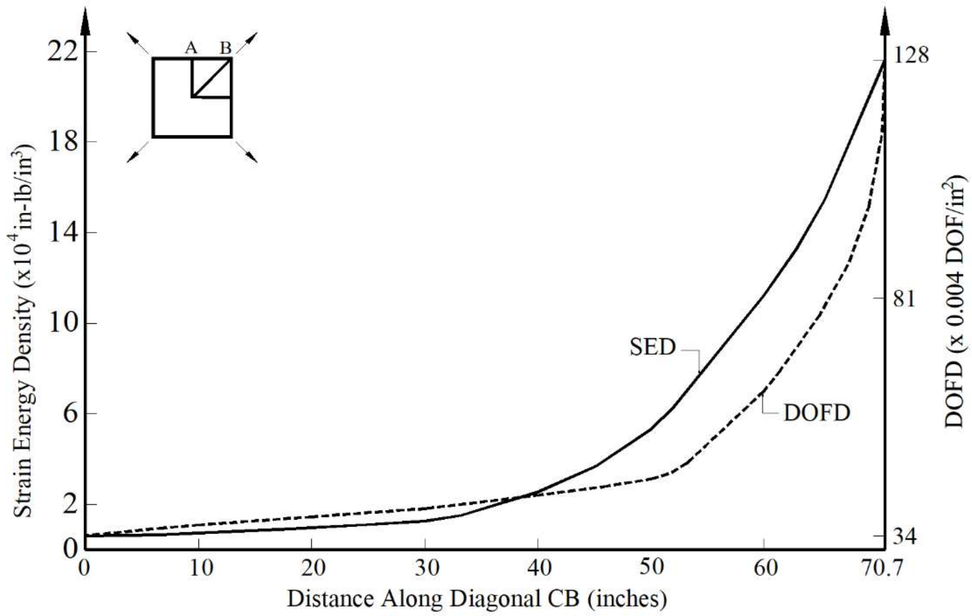


Figure 15. Strain energy density and degree of freedom density along edge AB using the new proposed guidelines.

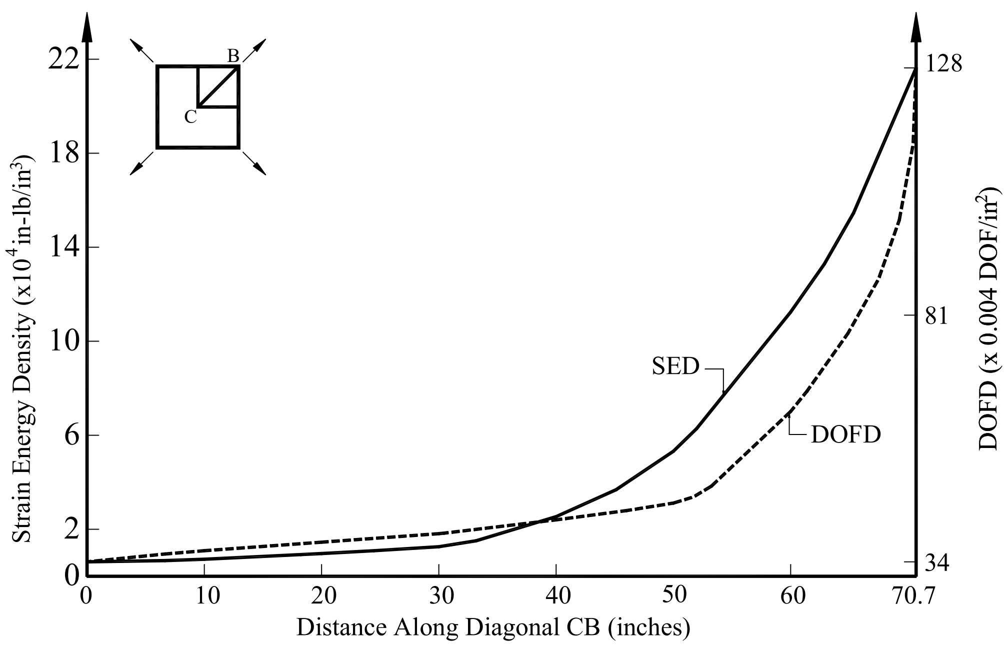


Figure 16. Strain energy density and degree of freedom density along diagonal CB using the new proposed guidelines.

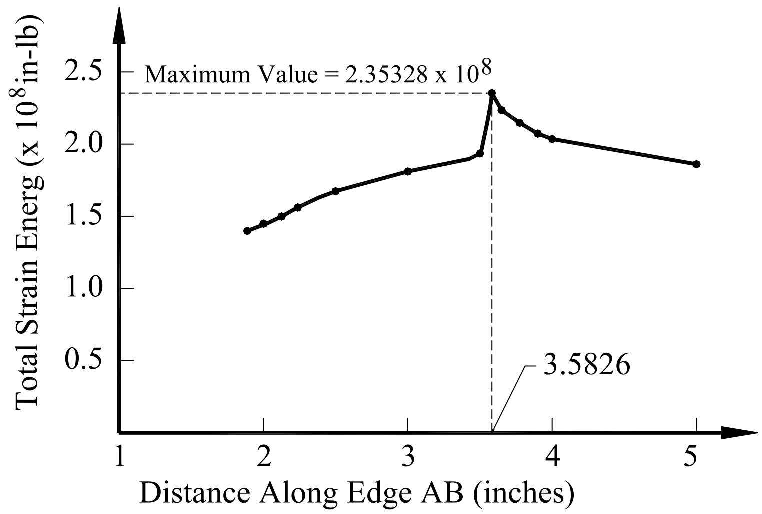


Figure 17. Total strain energy as a function of length ratio.

Table 1. Segment lengths along edge AB and diagonal CB.

Segment lengths along edge AB (inches)	Segment lengths along diagonal CB (inches)
36.0484	50.9801
10.0621	14.2300
2.8086	3.9720
0.7839	1.1086
0.2189	0.3096
0.0610	0.0863
0.0171	0.0242

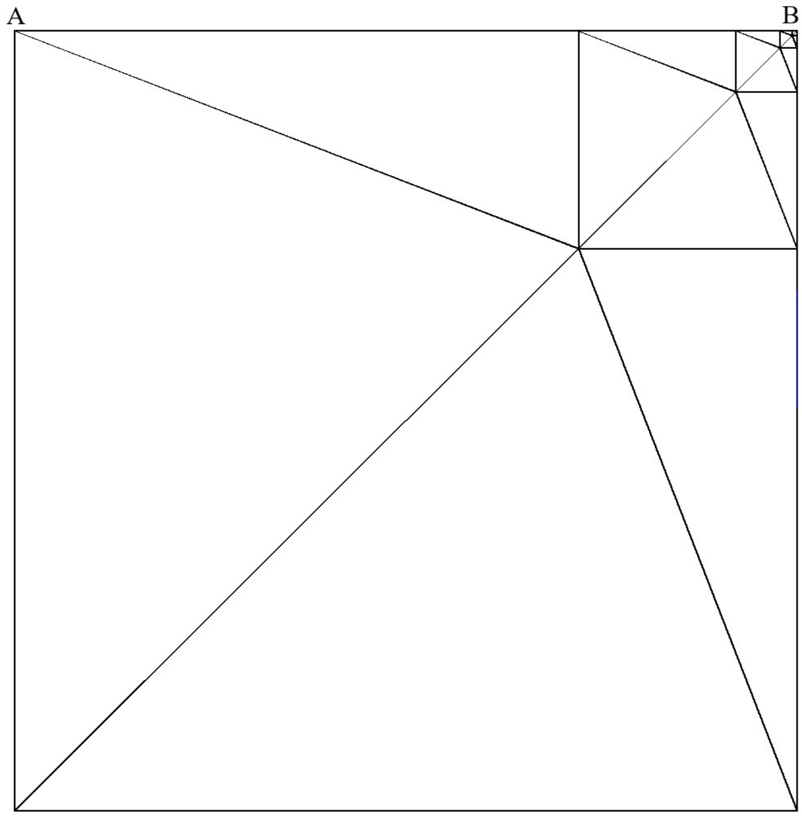


Figure 18 Optimum mesh using the proposed guidelines.

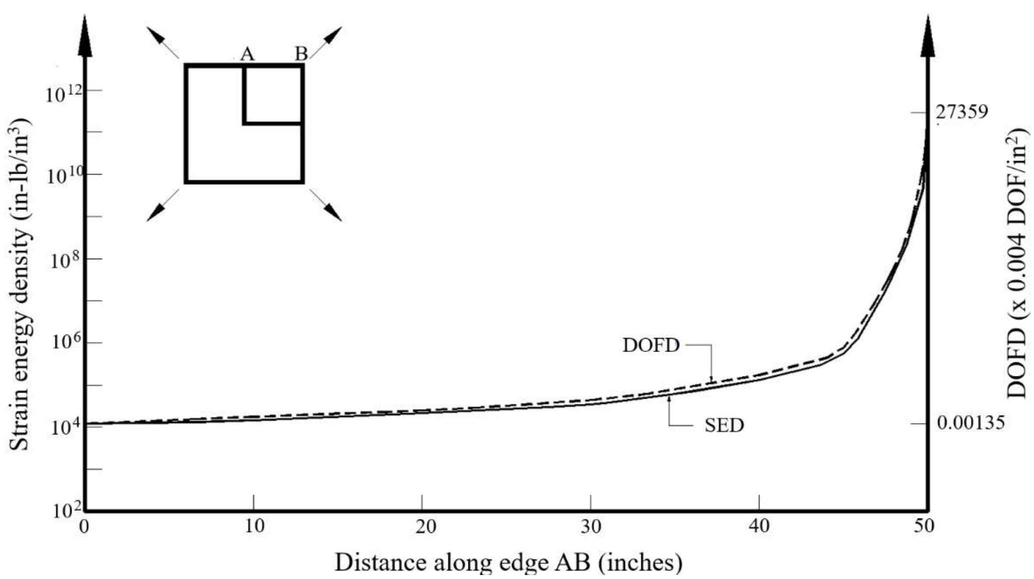


Figure 19. Strain energy density and degree of freedom density along edge AB using the proposed guidelines.

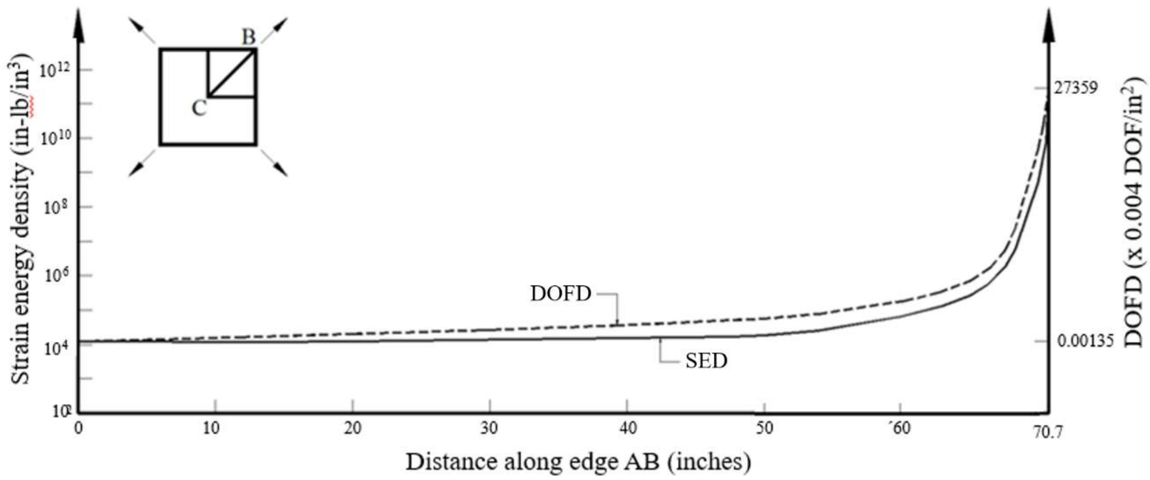


Figure 20. Strain energy density and degree of freedom density along diagonal CB using the proposed guidelines

Comparison and Discussion of Results

The total strain energies and maximum displacements for different optimal meshes obtained using various methods are presented in Table 2. The table shows that the total strain energies of the optimal meshes based on Turcke and McNeice's guidelines, Turcke's mathematical programming technique, and Shephard's contouring method (Shephard, 1979) are approximately 48.0%, 50.0%, and 53.8%, respectively, of those obtained using the proposed guidelines. When the X-direction is considered horizontal and the Y-direction vertical, the maximum displacements in both directions, as shown in Table 2, exhibit a trend similar to that observed for the total strain energies.

Conclusions

Under external loading, when the load is concentrated at a single node of the finite element mesh of a specific structure, the literature has shown that numerical solutions obtained using various guidelines for different optimal meshes—each developed by different researchers based on a single-variable approach—are generally suboptimal. To overcome this limitation, the present study proposes new guidelines for optimal mesh generation based on a two-variable approach. Case studies are subsequently employed to illustrate the following two conclusions:

1. If the variation of the degree of freedom density (DOFD) matches that of the strain energy density (SED) along any selected path in a mesh, the condition for an optimal finite element solution is considered satisfied. Although this condition may not be fully attainable in practice, it remains a useful criterion for evaluating the quality of finite element solutions.

2. Numerical results indicate that the newly proposed two-variable guidelines for optimal finite element solutions enable analysts to achieve substantially improved results. This improvement stems from the guidelines' ability to accurately capture the essential features of mesh optimization

Table 2. Total strain energies and maximum displacements of different optimal meshes obtained by various methods

	Total strain energy, ($\times 10^8\ lb/in$)	Maximum displacement in the X-Direction (in)	Maximum displacement in the Y-Direction (in)
Authors' guidelines	2.3533 (100%)*	11.050 (100%)	11.050 (100%)
Turcke and McNeice's guidelines	1.1299 (48.0%)	6.393 (57.8%)	6.393 (57.8%)
Turcke's mathematical programing technique	1.1756 (50.0%)	6.649 (60.2%)	6.649 (60.2%)
Shephard et al.'s contouring method	1.2671 (53.8%)	7.167 (64.9%)	7.167 (64.9%)

References

Melosh R. J., and Marcal, P. V., "An Energy Basis for Mesh Refinement of Structure Continuum," International Journal of Numerical Methods in Engineering, Vol. 11, No. 7, pp. 1013-1092, 1977

Oliveria, E. R. A., "Optimization of Finite Element Solutions," Proceedings of the third. Conference on Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, Ohio, October 1973

Shephard, M. S., "Finite Element Grid Optimization-A Review," Rensselaer Polytechnic Institute, Troy Book Company, New York, 1979

Shephard, M. S., Gallagher, R. H., and Abel, J. F., "Experiences with Interactive Computer Graphics for the Synthesis of Optimal Finite Element Meshes," in Finite Element Grid Optimization, ASME Special Publication PVP-38, ed. Shephard, M. S., and Gallagher, R. H., New York, 1979

Turcke, D. J., "Optimum Mesh Configuration in the Finite Element Method," Dissertation submitted to the Department of Civil Engineering of University of Waterloo in partial fulfillment of the requirements for the Degree of Doctor of Philosophy, Canada, July 1974

Turcke, D. J., "Characteristics of Piecewise Approximations in Numerical Analysis," in Finite Element Grid Optimization, ASME Special Publication PVP-38, ed. Shephard, M. S., and Gallaaher, R, H., 1979

Turcke, D. J., and McNeice, G. M., "Guidelines for Selecting Finite Element Grids Based on an Optimization Study," Computer and Structures, Vol. 4. pp. 499-519. 1974